Basics of MHD

{ N.B.: Read Kulsrud, Chapt. 3, 4 }

→ MHD Equations → Eulerian Fluid

i)
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$
(Continuity)

ii)
$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} \right) = -\underline{\nabla} P + \frac{\underline{J} \times \underline{B}}{c} + \underline{f}_{body}$$
(momentum balance)

[frequently $\underline{f}_{body} = \rho \underline{g}$]

iii)
$$\frac{d}{dt} S = \frac{\partial S}{\partial t} + \underline{v} \cdot \underline{\nabla} S = 0$$

(isentropic fluid)

$$S = C_v \ln (P/\rho^{\gamma})$$
↑
entropy

[frequent form of equation of state]

iv)
$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \left(\eta \underline{J} + \frac{\underline{J}}{\sigma} \right)$$
(Ohms Law)

[resistivity η is usually most significant dissipation]

ideal MHD

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = 0$$

and

(5) $$\underline{\nabla}\cdot\underline{B} = 0$$

(6) $$\underline{\nabla}\times\underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}$$

(7) $$\underline{\nabla}\times\underline{B} = \frac{4\pi}{c}\underline{J}$$

} from Maxwell's Eqns. neglecting displacement current

→ Meaning, Restriction, Validity

- MHD is simplest, closed, self-consistent plasma model, and the most heavily exploited for dynamical modelling.

· Variants : Reduced MHD → strong $\underline{B}_0$ (tokamaks)
  2D MHD
  E MHD → stationary ions (ICF)
  FLR MHD } → MHD + additional effects (MFE, space)
  Reduced Braginsky }
  hybrid → { bulk - MHD ; hot species - kinetic (i.e. α's energetics) }
  ⋮

· MHD - Eulerian

"fluid element" ⇒ "glue"

fluid element   fixed observer

here "glue" → collisions

→ MHD is: – strongly collisional
– low frequency
– large scale

i.e. frequencies relevant:

$$\omega \ll \Omega_{e,i};\ \omega_{p_{e,i}};\ \nu_{e e, e i};\ \nu_{ii};\ \omega_{*e,i}$$

etc.

scales relevant:

$$L \gg \lambda_{D_{e,i}},\ \rho_{e,i},\ c/\omega_{p_{e,i}},\ l_{mfp_{e,i}}$$

$$l_{mfp} < L$$

and

collisions isotropise, equilibrate $\underline{\underline{P}}$.

$$\left(\text{i.e.}\quad \underline{\underline{P}} \sim \int d^3v\ \tilde{v}_i \tilde{v}_j\ f(\underline{x}, \underline{v}, t) \right)$$

▷ Some Specific Points:

– re: continuity ①;

$$\rho = m_i n_i + m_e n_e$$

i.e. (ions control fluid inertia)

{ total density
{ ion dominated

- re: momentum balance ②;

$\triangleright$ $\underline{V} = \left(\int d^3v_i \, m_i \, \underline{v}_i \, f_i + \int d^3v_e \, m_e \, \underline{v}_e \, f_e \right) / \rho$

i.e. (ions control flow — $\rho \frac{d\underline{v}}{dt}$)

> where has $\underline{E}$ gone? $\rightarrow$ $L \gg \lambda_D$ $\rightarrow$ quasi-neutrality

$$\rho_i \frac{d\underline{v}_i}{dt} = n_i q_i \underline{E} + n_i q_i \frac{\underline{v}_i \times \underline{B}}{c} + \cdots$$

$$\rho_e \frac{d\underline{v}_e}{dt} = -n_e q_i \underline{E} - n_e q_i \frac{\underline{v}_e \times \underline{B}}{c} + \cdots$$

fadd: $\rightarrow 0$ $\qquad \rightarrow \frac{\underline{J} \times \underline{B}}{c}$

$n_i = n_e$ (quasi-neutrality) $\qquad$ (Lorentz force term in momentum balance)

note also: $\rho_i, \rho_e \rightarrow \rho$

> re-writing the $\underline{J} \times \underline{B}$ force:

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{(\underline{\nabla} \times \underline{B}) \times \underline{B}}{4\pi} = -\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \underline{\nabla} \underline{B}}{4\pi}$$

So can write:

$$\rho \frac{d\underline{v}}{dt} = -\underline{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}$$

magnetic pressure (field energy density)

magnetic tension

a) What / why "Magnetic Tension"?

$$\underline{B} = B\,\hat{b} \qquad B = |\underline{B}|, \quad \hat{b} = \underline{B}/B$$

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = B\,\hat{b}\cdot\underline{\nabla}\,(B\hat{b})$$

$$= \underset{(1)}{B^2\,\hat{b}\cdot\nabla\hat{b}} + \underset{(2)}{\underline{\hat{b}}\,\hat{b}\cdot\nabla}\left(\frac{B^2}{2}\right)$$

$\underline{\hat{b}}\cdot\underline{\nabla}\,\underline{\hat{b}} \rightarrow$ curvature of $\hat{b}$
(i.e. rate of change of $\hat{b}$ along itself)
$\equiv d\hat{b}/ds$

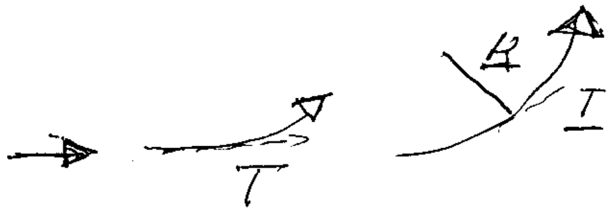
n.b. in general: curve: $\underline{x}(t)$

tangent: $\underline{T} = d\underline{T}/ds$

$(ds^2 = d\underline{x}\cdot d\underline{x})$ $\quad s \equiv$ distance along curve curvature $\underline{K} = \frac{d\underline{T}}{ds} = \frac{d\underline{T}/dt}{ds/dt} = \frac{\dot{\underline{T}}}{|\underline{V}|}$

Now: $\underline{K} = \hat{\underline{b}} \cdot \nabla \hat{b}$ → points in direction of turning of $\hat{b}$, orthogonal to $\hat{b}$

i.e.



$\underline{K} = d\underline{T}/ds$

∴ $\underline{K} = +\frac{\hat{n}}{R_c}$ $R_c \equiv$ radius of curvature

as curved field line suggests "tension" → "magnetic tension".

b.) What about ② ?

But $\underline{J} \times \underline{B} \perp \underline{B}$ yet $\nabla\left(\frac{B^2}{8\pi}\right)$ can have component along $\underline{B}$??

→ recombining total $\underline{J} \times \underline{B}$ gives:

$$-\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + B^2\frac{\hat{b}\cdot\underline{\nabla}\hat{b}}{4\pi} + \hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)$$

$$= -\nabla_\perp\left(\frac{B^2}{8\pi}\right) \cancel{-\hat{b}\hat{b}\cdot\nabla\left(\frac{B^2}{8\pi}\right)} \cancel{+\hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)} + B^2\frac{\hat{b}\cdot\nabla\hat{b}}{4\pi}$$

$$\Rightarrow \boxed{\underline{J}\times\underline{B} = -\underline{\nabla}_\perp\left(\frac{B^2}{8\pi}\right) + B^2\frac{\hat{b}\cdot\underline{\nabla}\hat{b}}{4\pi}}$$

3) $dE = đQ - p\,dV$ (Thermo)

$C_v\,dT = T\,dS - p\,dV$

$\begin{cases} đQ = T\,dS \\ dE = C_v\,dT \end{cases}$ (normalized)

$V = 1/\rho \qquad dV = -\,d\rho/\rho^2$

$$C_v\frac{dT}{T} = dS + \frac{d\rho}{\rho}$$

$$\Rightarrow \ln T = \frac{S}{C_v} + \ln\rho^{1/C_v}$$

$$\therefore \boxed{S' = C_v\ln\left(T/\rho^{1/C_v}\right)}$$

$p = \rho T$

$$\Rightarrow \quad s = c_v \ln\left(P/\rho^{(c_v+1)/c_v}\right)$$
$$= c_v \ln\left(P/\rho^{\gamma}\right)$$

$\gamma = 5/3$, ideal gas

$\left(c_v = 3/2 \text{ (normalized)}\right)$

$$\frac{ds}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(P/\rho^{\gamma}\right) = 0$$

i.e.
$$\boxed{\frac{\partial}{\partial t}\left(P/\rho^{\gamma}\right) + \underline{V}\cdot\underline{\nabla}\left(P/\rho^{\gamma}\right) = 0}$$

eqn. of state

perfect $\begin{cases}\text{homogeneity} \\ \text{stationarity}\end{cases}$ $\left(P/\rho^{\gamma} = \text{const.}\right)$

– "adiabatic equation of state"

4) Ohm's Law – most sensitive part of MHD (since controlled by electrons)

MHD variants differ primarily in Ohm's Law

- Hall MHD $\rightarrow$ Hall term
- EMHD $\rightarrow$ electron inertia
- Braginsky / drift MHD $\rightarrow$ ∇P terms

⋮ etc., etc.

⋮

Ohm's Law → subtract moments on electron equations → electrons $\left(\underline{J} = n e (\underline{V}_i - \underline{V}_e) \right)$

Simple resistive MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

$\sim \nu_{ei}$ → momentum transfer to ions ...

ideal MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = 0$$

5, 6, 7: Only 1 approximation:

$$\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t}$$

$|\partial \underline{E}/\partial t| << |\underline{J}|$ → condition on ω?

→ $\omega \frac{V B}{c} \overset{?}{<<} \frac{k}{c^{-1}} B$

⇒ $|\underline{V}| (\omega/k)/c^2 << 1$ is condition on ω.

→ Skeptic: "Does it Hang Together"?

i.e. is electric force negligible?

$$\rho \frac{d\underline{v}}{dt} = n q \underline{E} + \cdots\cdots$$

and $q \neq 0$, as

$$n q = \frac{\nabla \cdot E}{4\pi}$$

$$\underline{E} = -\frac{\underline{v} \times \underline{B}}{c}$$

so

$$n q \underline{E} = +\frac{(\underline{v} \times \underline{B})}{c} \, \underline{\nabla} \cdot \left(\frac{\underline{v} \times \underline{B}}{c} \right) \neq 0 \quad !$$

it

$$\sim \frac{v^2}{c^2} B^2 k$$

$$\sim \frac{v^2}{c^2} (\underline{J} \times \underline{B}) \longrightarrow \text{negligible.}$$

Thus, yes indeed it does!

→ Putting it together:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J} \quad , \quad \underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

⇒ the induction equation for $\underline{B}$ evolution ...

$$\boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}}$$

- with momentum equation, defines MHD as problem of 2 coupled fluid fields (vector) – $\underline{V}(\underline{x},t)$, $\underline{B}(\underline{x},t)$ evolving simultaneously

↓

- useful and instructive to re-write induction equation

$$\nabla \times \underline{V} \times \underline{B} = -\underline{V} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

so

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}.$$

This brings us to ......

⇒ What Does "MHD", as a system, really mean ....?

this is answered most clearly for the case of
incompressible MHD ....

$\underline{\nabla} \cdot \underline{V} = 0$ → defines equation of state
→ sets P_{total} field

$\omega/k \ll C_s, V_{MS}$)
sound → magnetosonic

$$\nabla \cdot \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla} \underline{V} = -\frac{\underline{\nabla}}{\rho}\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \underline{\nabla} \underline{B}}{4\pi\rho} \right\}$$

$$\frac{d\rho}{dt} = -\rho\, \underline{\nabla} \cdot \underline{V} = 0$$

so ρ → constant ρ_0 (can relax to slow variation)

$$\nabla^2 \left[\left(P + \frac{B^2}{8\pi}\right) / \rho_0 \right] = \underline{\nabla} \cdot \left(\frac{\underline{B} \cdot \underline{\nabla} \underline{B}}{4\pi\rho_0} - \underline{V} \cdot \underline{\nabla} \underline{V} \right)$$

↑
total pressure

a la' Poisson's equation:

$$\frac{P + B^2}{8\pi} = -\int \frac{d^3x'}{4\pi |x - x'|} \left\{ \underline{\nabla} \cdot \left(\frac{\underline{B} \cdot \underline{\nabla}' \underline{B}}{4\pi\rho_0} - \underline{V} \cdot \underline{\nabla}' \underline{V} \right) \right\}$$

solves for: P_{tot} field → eliminates eqn. state.

$p^* = p_{tot}$

so

$$\boxed{\begin{aligned} \frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} &= -\underline{\nabla}\left(\frac{p^*}{\rho_0}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho_0} \\ \frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} - \eta\,\nabla^2\underline{B} &= \underline{B}\cdot\underline{\nabla}\,\underline{V} \end{aligned}}$$

with $\nabla\cdot\underline{V} = 0$, constitute equations of incompressible MHD.

Rather clearly, this system is one of <u>two</u>, dynamically coupled, evolving vector fields $\underline{V}(\underline{x},t)$, $\underline{B}(\underline{x},t)$.

→ <u>Compressible</u> MHD is really a problem in <u>3</u> fields, two of which are vectors

<u>i.e.</u>

$$\begin{cases} \underline{V}(\underline{x},t) & \to \text{fluid velocity} \\ \underline{B}(\underline{x},t) & \to \text{magnetic field} \\ S(\underline{x},t) & \to \text{entropy} \Rightarrow \text{energy density} \end{cases}$$

<u>i.e.</u> scalar equation of state provides 3^{rd} field.

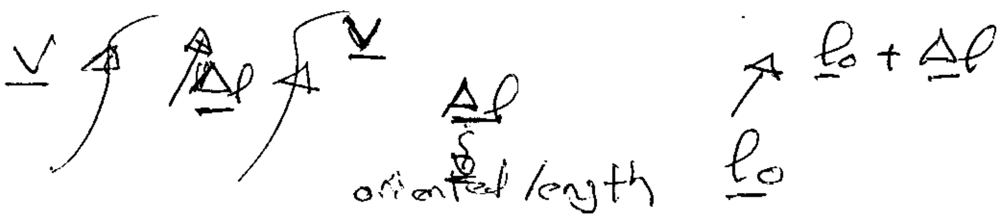
⇒ Key Question: How closely coupled are $\underline{V}$, $\underline{B}$?

⇒ the key physics element in MHD ......

⇒ $\boxed{\text{Frozen-in Law, Flux Freezing}}$

I) Frozen-in Law

- consider a (for the moment, passive) vector field:
  - frozen into flow $\underline{V}(\underline{x}, t)$
  - consisting of oriented, flexible strands



How does $\underline{\Delta l}$ evolve?

in dt, $\quad d(\underline{\Delta l}) = (\underline{V}(\underline{l_0} + \underline{\Delta l}) - \underline{V}(\underline{l_0}))\, dt$

$$= \underline{\Delta l} \cdot \nabla \underline{V} \quad dt$$

$$\therefore \quad \frac{d(\underline{\Delta l})}{dt} = \underline{\Delta l} \cdot \nabla \underline{V}$$

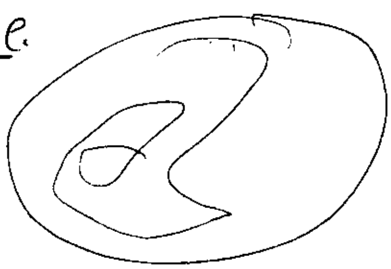
i.e. $\dfrac{d}{dt}\underline{\Delta \ell} = \underline{\Delta \ell}\cdot\underline{\underline{S}}$

$$\begin{cases} \dfrac{d}{dt}(\Delta \ell)_i = \underline{\Delta \ell}_j \cdot S_{ij} \\ S_{ij} = \dfrac{1}{2}\left(\dfrac{\partial V_i}{\partial x_j} + \dfrac{\partial V_j}{\partial x_i}\right) \end{cases} \Rightarrow \text{strain rate tensor}$$

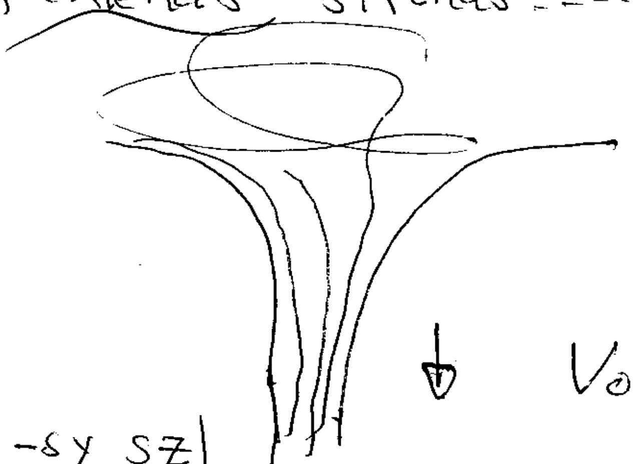
says that $\Rightarrow$ $\underline{\Delta \ell}$ strands orient along strain

$\Rightarrow$ strain extends strands ....

i.e.



$\longrightarrow$ siphon flow



$\underline{V} = V_0\left(-\dfrac{sx}{2}, -\dfrac{sy}{2}, sz\right)$

plausible to say that $\underline{\Delta \ell}$ "frozen onto" the flow.

Now, if $\eta \to 0$, ... in MHD

$$\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\;\underline{\nabla}\cdot\underline{V}$$

$$-\underline{\nabla}\cdot\underline{V} = +\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\frac{\partial}{\partial t}\underline{B} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \frac{d\underline{B}}{dt} = \underline{B}\cdot\underline{\nabla}\,\underline{V} + \frac{\underline{B}}{\rho}\frac{d\rho}{dt}$$

$$\frac{1}{\rho}\frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2}\frac{d\rho}{dt} = \frac{\underline{B}}{\rho}\cdot\underline{\nabla}V$$

$$\therefore \boxed{\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{\underline{B}}{\rho}\cdot\underline{\nabla}\,\underline{V}}$$

→ $\underline{B}/\rho$ obeys same equation as $\underline{\Delta l}$!

→ $\underline{B}/\rho$ is frozen into flow field $\underline{V}(\underline{x},t)$

Note: → $\underline{B}/\rho$ is **not** passive → due $\underline{J}\times\underline{B}$ force

→ $\underline{B}$ determines flow, while frozen into it!
(essence of coupling problem)

> For $\underline{\nabla} \cdot \underline{V} = 0$, $\underline{B}$ frozen in

> if $\eta \neq 0$, freezing in is broken ....

i.e. $\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) - \frac{\eta}{\rho} \nabla^2 \underline{B} = \frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V}$

form of frozen evolution broken

> Observe: → this motivates attention to resistivity in MHD above other dissipations ν, χ, etc..

→ $\eta \Rightarrow \underline{B}$ diffusion $\sim \eta \nabla^2$

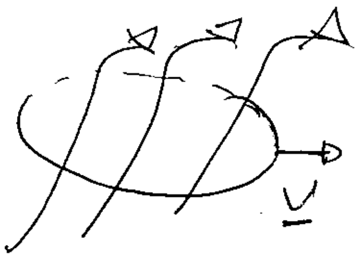
∴ decoupling of $\underline{V}$, $\underline{B}$ occurring on small scales

⇒ motivates 'magnetic reconnection' as study of singularity dynamics in MHD.

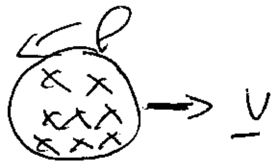
> A Word to the Wise: In modelling, describing complex dynamics in MHD (i.e. MHD turbulence, dynamos, etc.) always think carefully about frozen-in law ...

→ Closely Related: Flux Freezing

- consider flux thru surface in flow



$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{B}$$



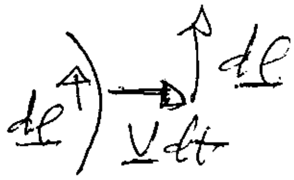
$$\underline{\Phi} = \int \underline{B} \cdot d\underline{s}$$

$$\frac{d\Phi}{dt} = \underbrace{\int d\underline{s} \cdot \frac{\partial \underline{B}}{\partial t}}_{①} + \underbrace{\int \frac{d\underline{s}}{dt} \cdot \underline{B}}_{②}$$

$$① = \int d\underline{s} \cdot \underline{\nabla} \times (\underline{V} \times \underline{B})$$

$$= \oint d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$

For ②



so $d\underline{s} = \underline{V} dt \times d\underline{\ell}$

↳ {change in $\underline{s}$ in dt.

$$dt \, (d\Phi_{②}/dt)$$

$$② dt = \int (\underline{V} dt \times d\underline{\ell}) \cdot \underline{B} = d\underline{\Phi}$$

$$\left.\frac{d\Phi}{dt}\right)_{②} = \int (\underline{V} \times d\underline{\ell}) \cdot B = -\int d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$

so $\dfrac{d\Phi}{dt} = \underbrace{① + ②}_{=0}$

so $\Rightarrow$ magnetic flux invariant $\Leftrightarrow$ cancellation

$\Rightarrow$ in absence of resistivity, flux thru surface in flow is invariant, or frozen in

$\Rightarrow$ no surprise: $\underline{B}$ frozen in $\Rightarrow$ Φ frozen in

$\Rightarrow$ analogue in hydro: Circulation (Kelvin's Thm.)

$$\Gamma_c = \oint \underline{v} \cdot d\underline{l} = \int d\underline{a} \cdot \underline{\omega} \qquad \omega = \nabla \times \underline{v}$$

In inviscid hydro, $(\nu \to 0)$ circulation Γ_c is conserved.

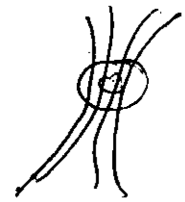
Exercise (#10 for Set 1): Prove this! Note relation between $\underline{\omega}$ equation and $\underline{B}$ eqn. Assume $\rho = \text{const.}$, $\underline{g} = 0$

Extra Credit: Discuss the extension to the case where $\rho \neq \text{const.}$

→ What Does "Freezing" Mean?

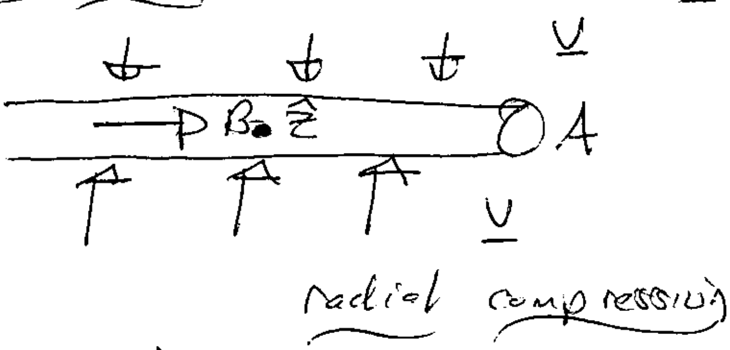
⇒ can relate field evolution in a flow to density evolution, since $\underline{B}/\rho$ is "frozen in".

Application: Star formation

→  (?)

Simple Cases → How does $\underline{B}$ change in a flow?

)



Cylinder of cross-sectional area A squeezed in radial flow. $\underline{B} \equiv B\,\hat{z}$

2 ways:

$$\frac{d}{dt}\left(B\hat{z}/\rho\right) = \frac{B\hat{z}\cdot\nabla\,\underline{V}}{\rho}$$

$$\underline{V} = v\,\hat{r}$$

$$\Rightarrow \underline{V} \perp \underline{B}$$

$$= 0$$

so $\underline{B}/\rho = \text{const}$

Now: $\rho A L = \text{const}$ so $B \sim A^{-1}$
$\rho \sim A^{-1}$
(L const.)

or

Flux Frozen: $BA = \Phi = \text{const.}$
$\rho A L = \text{const} = M$
L const.

$BA \sim \Phi_0$, $B \sim A^{-1}$
$\rho A \sim M_0$, $\rho \sim A^{-1}$
so $B \sim \rho^{(1)} \Rightarrow B/\rho \sim \text{const}!$

$V = V(z)\hat{z}$

i.) $\longrightarrow B\hat{z}$ i.e. stretch, 1D

here $\dfrac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V} \neq 0$, but easier to work with $\underline{B}$ than $\underline{B}/\rho$

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

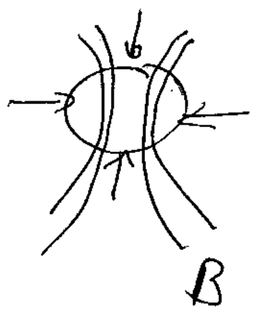
$$= B \frac{\partial}{\partial z} V(z) - B \frac{\partial V(z)}{\partial z}$$

$$= 0 \quad !$$

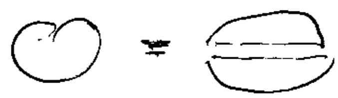
For ρ, $\quad \frac{d\rho}{dt} = -\rho\, \underline{\nabla} \cdot \underline{V} = -\rho \frac{\partial V_z}{\partial z}$

here $\underline{B}$ invariant, ρ changes

i.e. $\quad B \sim \rho^{(0)}$

ii.) 

collapsing sphere: $\underline{V} = v\,\hat{r}$



(i.e. $\Phi = 0$ total sphere)

consider hemispherical surface (i.e. mushroom cap)

$$\Phi \sim B R^2 \sim \text{const}$$

$$M \sim \rho R^3 \sim \text{const}$$

$$\Rightarrow \begin{array}{l} B \sim r^{-2} \\ \rho \sim r^{-3} \end{array} \quad \Rightarrow \quad B/\rho^{2/3} \sim \text{const.}$$

why the scaling? $\Leftrightarrow$ why of interest?

→ $\left\{\begin{array}{l}\text{"implosion"} \\ \text{gravitational collapse} \\ \text{equation of state}\end{array}\right\}$ problems sensitive to equation of state of material collapsing

If: $p \to p_{tot} = p + \frac{B^2}{8\pi}$

$p = p_0 (\rho/\rho_0)^\gamma$

collapse threaded by magnetic field

then natural to ask: Can one write $B^2 = B^2(\rho)$ and thus extend equation of state to encompass magnetic pressure contribution?

Proceed via flux-freezing!

$B \sim \rho^{2/3} \Rightarrow B^2 \sim \rho^{4/3}$

so p_{B^2} has "γ_{eff}" $= 4/3$. This resembles equation of state for degenerate gas (see Handouts I).

→ More on this in discussion of flux freezing and virial theorems ....

→ Pragmatic Question: Is flux 'frozen' during star formation? ↔ Does resistivity matter?

$\eta \sim \frac{4 \times 10^6}{T_{eV}^{3/2}}$ cm²/sec. (Spitzer)

start → collapse → protostar

$n \sim 1\,\text{atom/cm}^3$ ⟶ $\rho \sim 1\,\text{g/cm}^3$ ⇒ $n \sim 10^{24}_{atm}/\text{cm}^3$

but

$B/\rho^{2/3} \sim$ const

⇒ $B/B_0 \sim (10^{24})^{2/3} \sim 10^{16}$! huge amplification

so $B_0 \sim 10^{-6}$ G, characteristic of ISM

⇒ $B \sim 10^{10}$ G in protostar

∴ $P_{B^2} \sim 10^{19}$ erg/cm³ $(P_{B^2} \sim B^2/8\pi)$

but P_{Th} for normal star $\sim 10^{14}$ erg/cm³

$P_{B^2} \gg P_{Th}$?? ⇒ clearly flux-freezing is <u>bad</u> assumption

→ In terms of time scales:

$$\frac{\partial B}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) + \eta \nabla^2 B$$

① ② ③

$$\frac{1}{T_{collapse}} \sim \frac{1}{T_{dynamic}} + \frac{\eta}{L^2}$$

$\S$ $1/T_{diffn.}$

3 scales, 2 balance

i.e. ① & ② ③ negligible

① & ③ ② negligible.

if $T_{collapse} \ll T_{diffn}$ → $\{$ flux frozen, OK

$T_{collapse} \gg T_{diffn}$ → $\{$ must consider diffusion freezing invalid

N.B.: In star formation, $T_{coll.} \ll T_{diffn}$

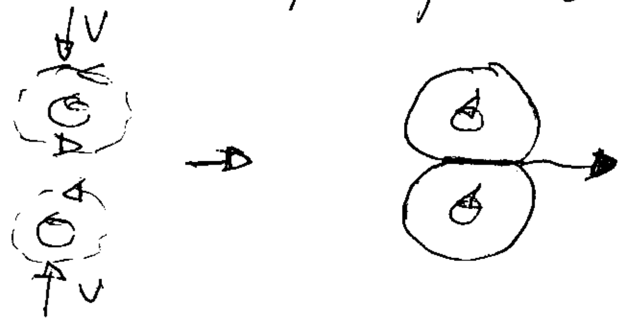
<u>but</u> ISM has large neutral component.

Plasma-neutral <u>drag</u> sets dissipation
→ Ambipolar diffusion.

→ Breakdown of Flux Freezing – Magnetic Reconnection ?!

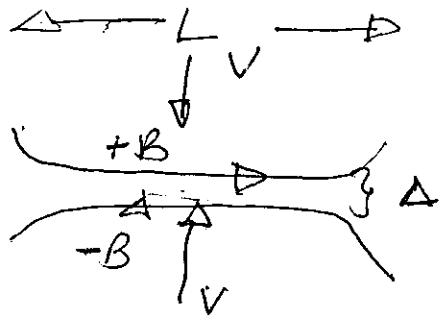
Simple Example : Sweet – Parker Problem
(re-visit later)

→ consider two cylinders of plasma, carrying current ⊥ plane, brought together



reconnection layer (Δ small → η/Δ^2 significant)
↓
flux freezing broken

⇒ consider layer



2 plasma slabs brought together at V

$\Delta < L$

What Happens
Stationary Solution Possible?

$\nabla \cdot \underline{v} = 0$ $$\frac{d\underline{B}}{dt} = \underline{B} \cdot \nabla \underline{v} + \eta \nabla^2 \underline{B}$$

$$S_{ij}^{(0)} = \begin{pmatrix} 0, & 0 \\ 0, & -V\delta(y) \end{pmatrix}$$

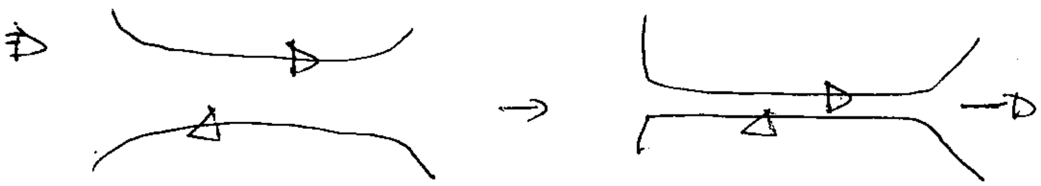
↓
rate-of-strain tensor

→ singularity !

tip-off of small scale generation in $\underline{B}$
⇒ resistive diffusion, breaking of freezing in . .

i.e. for stationary solution,

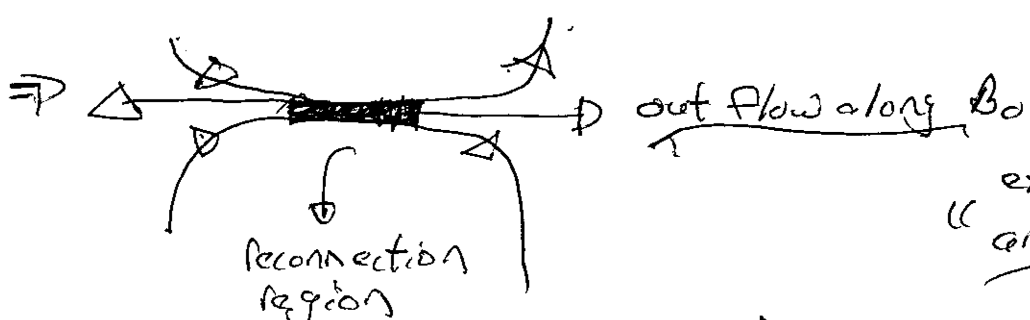
$$-\frac{\underline{B}\cdot\underline{\nabla}\,\underline{V}}{\eta} = \nabla^2\underline{B}, \qquad \underline{\nabla}\cdot\underline{V}=0$$



Δ small

$\nabla(B^2/8\pi)$ increasing

fluid expelled ?

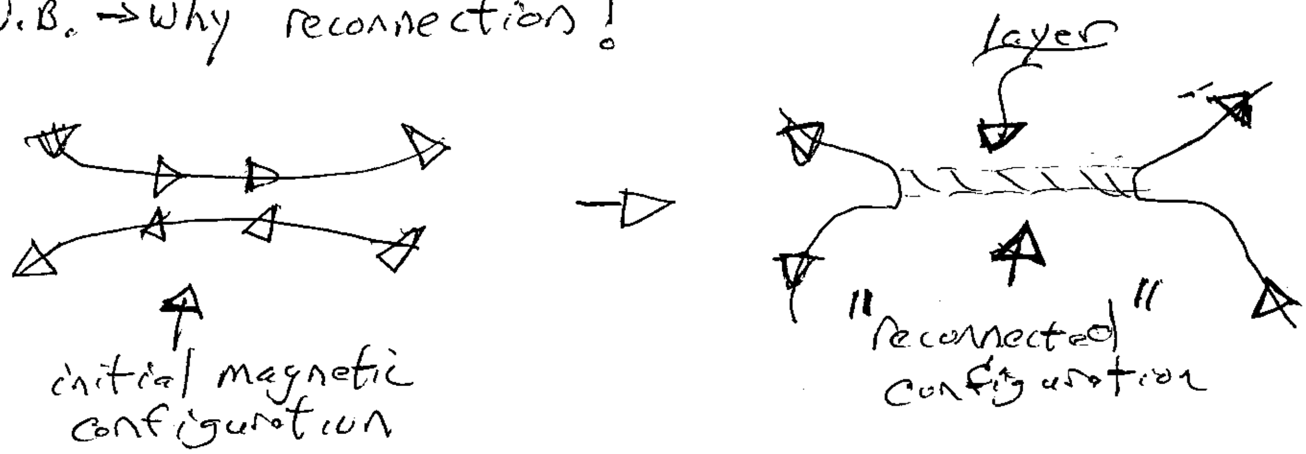


(large resistive dissipation)

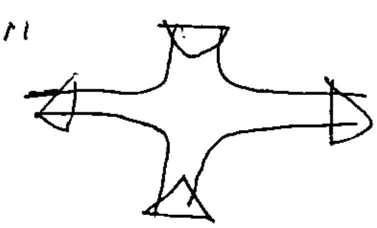
expect opposed fields "annihilate"

N.B. A particlular value is required for stationarity

N.B. → why "reconnection"?



→ Flow is "stagnation point"



» How Calculate? → Match In-Flow → Out-Flow
(S-P. is a great Back-of-Envelope ...)

Conserved: ① – mass $(\underline{\nabla}\cdot\underline{V} = 0)$

② – momentum in $\hat{x}$ direction (symmetry)

③ – energy balance, →
→ rate of field 'delivery' to reconnection region
MUST BALANCE
→ rate of ohmic dissipation

① extent in $\hat{x}$, extent in $\hat{y}$

$$\underset{\text{inflow}}{\rho_0 V L} = \underset{\text{outflow}}{\rho_0 V_0 \Delta}$$

$$VL = V_0 \Delta$$

$$V = V_{out}\,\Delta / L$$

2) $$\rho_0\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\underline{B}}{4\pi}$$

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla}\left(\frac{V^2}{2}\right) + \underline{V}\times\underline{\omega}$$

symmetry: $$0 = \underline{\nabla}\left(P + \frac{B^2}{8\pi} + \frac{\rho_0 V^2}{2}\right)$$

modified Bernoulli Eqn.

$\underline{a}$ → $V = 0$, B finite

$\underline{a}$ $\underline{b}$

$\underline{b}$ → $V = V_{out}$, $B \to 0$

$$\left(\frac{B^2}{8\pi} \ll P\right)$$

so $\quad p + \frac{B^2}{4\pi} + \frac{\rho_0 V^2}{2} = \text{const.}$

$$p + \frac{B^2}{8\pi} = p + \frac{\rho_0 V_{out}^2}{2}$$

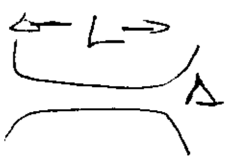
$$V_{out} = B^2/4\pi\rho_0 = V_A^2$$

$\hookrightarrow$ Alfven speed

$V_{out} = V_A$

$V = V_A \frac{\Delta}{L}$

specifies "speed" V, in terms Δ.

3) 

Energy balance

$\Rightarrow$

$$\begin{pmatrix}\text{Rate of Magnetic} \\ \text{Energy Inflow}\end{pmatrix} = \begin{pmatrix}\text{Rate of Ohmic} \\ \text{Dissipation, net}\end{pmatrix}$$

$P_{OH} = \frac{J^2}{\sigma}\Delta L$ so $\quad \dot{E}_{OH} = \frac{J^2}{\sigma} L\Delta$

$$= \left(\frac{c}{2\pi}\right)^2 \frac{B^2}{\Delta^2}\frac{L\Delta}{\sigma}$$

$\nabla \times B = \frac{4\pi}{c} J$

$2B = \frac{4\pi}{c} J\Delta$

$$P_{in} = 2\left(B^2/8\pi\right) VL = \dot{E}_{in}$$

balance $\Rightarrow \cancel{2}\left(\frac{\cancel{B^2}}{\cancel{8\pi}}\right) V \cancel{L} = \frac{c^2}{\cancel{4\pi}\,\sigma}\frac{\cancel{B^2}}{\Delta^2}\cancel{L}\,\Delta$

$$V = \left(\frac{c^2}{\pi\sigma}\right)/\Delta \sim \frac{\eta}{\Delta}$$

$$\frac{c^2}{4\pi\sigma} \equiv \eta \left(\sim \frac{L^2}{T}\right)$$

$V = V_A \Delta / L$

$V = \eta / \Delta$

$\Rightarrow \quad \frac{\Delta}{L} = \left(\frac{\eta}{L V_A} \right)^{1/2} = \left(1 / R_m \right)^{1/2}$

and

$V = V_A / \sqrt{R_m}$

$R_m = \frac{VL}{\eta} \equiv$ Magnetic Reynolds # (here with $V = V_A$)

$\Rightarrow$ Punch Line: (for large R_m)

① – layer is thin $\quad \frac{\Delta}{L} \sim 1/\sqrt{R_m}$

– speed is faster than η/L, slower than V_A $\Big\} \quad V \sim V_A / \sqrt{R_m}$

② flow pattern is a/a' stagnation $\Rightarrow$ $\Big\{$ ejection from reconnection layer at V_A

Moral of this story:

→ freezing-in violated when flows brings opposing $\underline{B}$ into contact.

$\Rightarrow$

→ generates singularities ⇒ thin current layers, which alter critical magnetic topology

⇒ "magnetic reconnection", "tearing", etc..